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AN EXAMPLE CONCERNING THE NONLINEAR POINTWISE ERGODIC THEOREM

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ABSTRACT
It is shown that the pointwise ergodic theorem for Markov operators in L_1, having a finite invariant measure, fails to extend to the case of nonlinear operators.

1. Introduction

Beginning with the work of Baillon [B 1], [B 2], a lot of effort has recently been devoted to the study of ergodic theorems for nonlinear operators, see [K]. However, all papers on this subject deal only with questions of weak and strong convergence, while the problem of almost sure convergence, which plays a dominant role in the linear ergodic theorems, has been ignored.

The simplest classical pointwise ergodic theorem for linear operators is the theorem of Hopf [H]: If T is a linear operator in L_1 of a probability space with $\| T \| = 1$, which is order preserving and leaves the constant 1 fixed, then the averages $A_n f = n^{-1}(f + Tf + \cdots + T^{n-1}f)$ converge a.e. for all $f \in L_1$.

We shall show that this theorem fails to extend to nonlinear T in a very strong sense.† We construct an order preserving T in L_1 with $T0 = 0$ which is nonexpansive in L_p for $1 \leqq p \leqq \infty$, and a bounded function $f \geqq 0$ such that $A_n f$ diverges on a set of positive measure. T is positively homogeneous and leaves the nonnegative constants fixed. $T^n g$ (and hence $A_n g$) does converge in L_p-norm for all $g \in L_p$ $(1 \leqq p < \infty)$ in our example.

Hopf's theorem makes more stringent assumptions than all other pointwise operator ergodic theorems. E.g., the ergodic theorems of Dunford–Schwartz [DS] and of Akcoglu [A] are more general. It therefore seems that the example

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† Recall that T is called nonexpansive in L_p if $\| Tf - Tg \|_p \leqq \| f - g \|_p$ holds for all f and g.

essentially eliminates all hopes for *general* pointwise nonlinear ergodic theorems. Of course, the possibility of positive results for specific classes of nonlinear operators remains.

Roughly speaking, our T will be an infinite dimensional product of speed limit operators, introduced by Krengel and Lin [KL] in the 1-dimensional case.

2. Construction of T

For $n = 1, 2, \ldots$, let Ω_n be the unit interval $[0, 1[$, $\mathcal{A}_n$ the Borel-σ-algebra, and μ_n the Lebesgue measure. $(\Omega, \mathcal{A}, \mu)$ denotes the product of these measure spaces. T will be given in terms of a sequence of functions $\varphi_n : \mathbf{R}^+ \to [0, 1]$, which are decreasing but not necessarily strictly decreasing. We assume $\varphi_n(x) = 0$ for $x \geqq 1$. Some of the arguments will be the same as in the 1-dimensional case. They will not be given in detail. Therefore, it may be helpful to study section 5 of [KL] first. It is largely independent of the rest of [KL].

Let $X_n(\omega)$ denote the n-th coordinate of $\omega \in \Omega$. A subset of Ω of the form $\{\omega : a_n \leqq X_n(\omega) < b_n (n = 1, \ldots, N)\}$ will be called N-interval in Ω. $\mathcal{F}$ denotes the class of functions on Ω which are nonnegative multiples of the indicator function of a finite disjoint union H of N-intervals for some N. We first define an operator $T_{(1)}$ on $\mathcal{F}$. Consider a fixed $f = \alpha 1_H$ in $\mathcal{F}$. There exist finitely many disjoint $(N-1)$-dimensional intervals J_s in $\Omega_2 \times \cdots \times \Omega_N$ with union $\Omega_2 \times \cdots \times \Omega_N$ and numbers

$$0 \leqq a_{s1} < b_{s1} \leqq a_{s2} < b_{s2} \leqq \cdots \leqq a_{sk} < b_{sk} \leqq 1$$

with

$$H \cap \{(X_2, \ldots, X_N) \in J_s\} = \bigcup_{i=1}^{k} \{(X_2, \ldots, X_N) \in J_s \text{ and } a_{si} \leqq X_1 < b_{si}\}.$$

(k can depend on s; if the left hand side is empty for some s, put $k = 0$). The integral equation

$$c_{si}^t = \int_0^t \varphi_1(b_{si} + c_{si}^x)\,dx$$

has a unique solution. Put $c_{si} = c_{si}^1$ and define the sequence d_{si} by backward induction:

$$d_{sk} = c_{sk} \quad \text{and} \quad d_{si} = \operatorname{Min}(c_{si}, d_{s,i+1} + (a_{s,i+1} - b_{s,i})),$$

$(i = k-1, k-2, \ldots, 1)$. Next put

$$H_{s,i} = \{(X_2, \ldots, X_N) \in J_s, a_{si} + d_{si} \leqq X_1 < b_{si} + d_{si}\},$$

and

$$T_{(1)}f = \alpha \sum_s \sum_{i=1}^{k} {}^1H_{s,i}$$

(The conditions on φ_1 imply $b_{si} + d_{si} \leqq 1$.)

Of course, for $f \in \mathcal{F}$ the value of N is not unique. It may be replaced by any larger value. But the definition of $T_{(1)}f$ does not depend on this.

Let $\mathcal{F}_1$ denote the set of functions on Ω which can be written in the form $f = \sum_{j=1}^{m} f_j$ with $f_j \in \mathcal{F}$ and

$$\{f_1 > 0\} \supset \{f_2 > 0\} \supset \cdots \supset \{f_m > 0\}.$$

Put

$$T_{(1)}f = \sum_{j=1}^{m} T_{(1)}f_j.$$

As in [KL], $T_{(1)}$ is order preserving, $T_{(1)}0 = 0$, and $T_{(1)}$ is L_1-nonexpansive in $\mathcal{F}_1$, i.e., $f, g \in \mathcal{F}_1$ implies $\| T_{(1)}f - T_{(1)}g \|_1 \leqq \| f - g \|_1$. As $\mathcal{F}_1$ is dense in $L_1^+(\mu)$, $T_{(1)}$ extends by continuity to $L_1^+(\mu)$. Finally, for $f \in L_1(\mu)$, put $T_{(1)}f := T_{(1)}f^+$. As in [KL], $T_{(1)}$ is nonexpansive in L_∞, too. It follows that $T_{(1)}$ is order preserving and nonexpansive in L_p $(1 \leqq p \leqq \infty)$. For $f \geqq 0$ one easily checks that $\int T_{(1)}f d\mu = \int f d\mu$. In terms of [KL], $T_{(1)}$ is a speed limit operator, leaving all coordinates but the first fixed and moving f in the direction of the first coordinate with speed limit φ_1 for one time unit. $T_{(n)}$ is defined in exactly the same way using the n-th coordinate and the speed limit φ_n. If $f \in \mathcal{F}_1$ depends only on the first N coordinates and $n > N$ holds, then $T_{(n)}f = f$. This follows because we may put $0 = a_{s1} < b_{s1} = 1$ above. Finally put, for such an f,

$$Tf = T_{(N)}T_{(N-1)} \cdots T_{(1)}f.$$

Again, this definition is independent of N. T again extends to L_1^+ by continuity and to L_1 by putting $Tf := Tf^+$, and T has the properties of $T_{(1)}$ mentioned above.

Put

$$B = \bigcup_{m=1}^{\infty} \{X_m \leqq 2^{-(m+2)}\}.$$

We claim that it is possible to determine the speed limits φ_m in such a way that $A_n 1_B$ fails to converge a.e.

First observe that $0 \leqq T^k 1_B \leqq 1$ and hence $0 \leqq A_n 1_B \leqq 1$ for all n. Put

$$f_* = \liminf A_n 1_B \quad \text{and} \quad f^* = \limsup A_n 1_B.$$

It follows from Fatou's lemma and from

$$\int A_n 1_B d\mu = \int 1_B d\mu \leqq \sum_{m=1}^{\infty} \mu(X_m \leqq 2^{-(m+2)}) = \sum_{m=1}^{\infty} 2^{-(m+2)} = \tfrac{1}{4}$$

that $\int f_* d\mu \leqq \frac{1}{4}$.

Let h_m be the indicator function of $\{X_m \leqq 2^{-(m+2)}\}$. Then $h_m \leqq 1_B$ implies $A_n h_m \leqq A_n 1_B$ for all n. Put

$$C_m = \{\omega : \exists n \geqq m \text{ with } (A_n h_m)(\omega) \geqq \tfrac{1}{2}\}.$$

It will be sufficient to determine φ_m in such a way that

$$\mu(C_m) \geqq 1 - 2^{-(m+2)}.$$

Then the intersection C of the sets C_m has measure $\geqq \frac{3}{4}$, and $f^* \geqq \frac{1}{2}$ holds on C. This then yields

$$\int f^* d\mu \geqq \tfrac{1}{2} \cdot \tfrac{3}{4} = \tfrac{3}{8} > \tfrac{1}{4} \geqq \int f_* d\mu.$$

As h_m depends only on the m-th coordinate, we have $Th_m = T_{(m)} h_m$, and this function again depends only on the m-th coordinate. Inductively, we obtain $T^j h_m = T^j_{(m)} h_m$ for all $j \geqq 0$, and hence $A_n h_m = n^{-1} \sum_{j=0}^{n-1} T^j_{(m)} h_m$.

h_m is the indicator function of $\{a_1 \leqq X_m < b_1\}$ with $a_1 = 0$ and $b_1 = 2^{-(m+2)}$. (We need only one set J_s coinciding with the full space $\Omega_1 \times \cdots \times \Omega_{m-1} \times \Omega_{m+1} \times \cdots \times \Omega_N$, so that the condition involving J_s can be deleted.) If c_1^t is the unique solution of the integral equation

$$c_1^t = \int_0^t \varphi_m(b_1 + c_1^x) dx$$

then $T^j_{(m)} h_m$ is the indicator function of $\{a_1 + c_1^j \leqq X_m < b_1 + c_1^j\}$. If φ_m is very small but strictly positive in $[0, 1[$, then the movement of the interval $[a_1 + c_1^j, b_1 + c_1^j[$ is arbitrarily slow. We can make sure that

$$\{2^{-(m+2)} \leqq X_m < 1\} \subset C_m,$$

and this is enough to conclude the proof.

Heuristically, this is rather obvious. For a formal argument put $\varepsilon = 2^{-(m+3)}$, and let $m = m_2 < m_3 < \cdots$ be an increasing sequence. Put $\varphi_m(x) = \varepsilon / m$ for $0 \leqq x < 3\varepsilon$, and $\varphi_m(x) = \varepsilon / m_k$ for $k\varepsilon \leqq x < (k+1)\varepsilon$ $(3 \leqq k < 2^{m+3})$. Write $c(t) = c_1^t$ and put $n_2 = m_2 = m$ and $n_{k+1} = n_k + m_{k+1}$ $(k \geqq 2)$.

Clearly $b_1 = 2\varepsilon$, $c(0) = 0$. The speed is ε / m for the first m time units. Hence $c(n_2) = \varepsilon$. Then the speed is ε / m_3 for m_3 time units. Hence $c(n_3) = 2\varepsilon$.

Inductively, one obtains $c(n_{k+1}) = k\varepsilon$ as long as the movement does not stop because we reach the end of the unit interval.

Consider $x \in [0, 1[$ with $k\varepsilon \leqq x < (k+1)\varepsilon$. For $j = n_k$, $[c(j), c(j) + 2\varepsilon[= [(k-1)\varepsilon, (k+1)\varepsilon[$ contains x. For $j = n_{k+1}$, the interval $[c(j), c(j) + 2\varepsilon[= [k\varepsilon, (k+2)\varepsilon[$ still contains x. It follows that $T^j_{(m)} h_m(\omega) = 1$ if $X_m(\omega) = x$ belongs to $[k\varepsilon, (k+1)\varepsilon[$ and $n_k \leqq j < n_{k+1}$. Assuming $m_{k+1} \geqq n_k$, we have $(A_{n_{k+1}} h_m)(\omega) \geqq \frac{1}{2}$. This completes the proof.

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